Entangled States and the Gravitational Quantum Well

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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

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Resumo

Neste trabalho estuda-se o entrelaçamento quântico em variável contínua de um sistema de duas partículas sujeitas à influência do campo gravitacional terrestre. Obtém-se uma descrição de espaço de fase do sistema bipartido através do cálculo da correspondente função de Wigner e verifica-se a presença de entrelaçamento por aplicação da generalização do critério Positive Partial Transpose para sistemas não-gaussianos. Examina-se também a influência da gravidade num possível protocolo de entrelaçamento quântico a ser partilhado entre estações sujeitas a diferentes valores do potencial gravitacional, com base na correlação de estados do poço gravitacional quântico.

Esta tese reflete o trabalho desenvolvido na Ref. [1].

Abstract

In this work, continuous variable entanglement is studied on a system of two particles under the influence of Earth's gravitational field. A phase-space description of the bipartite system is determined by calculating its Wigner function and the entanglement is verified by applying a generalization of the Positive Partial Transpose criterion for non-Gaussian states. The influence of gravity is also examined on an idealized entanglement protocol to be shared between stations at different potentials, based on the correlation of states of the gravitational quantum well.

This thesis is based on the work developed in Ref. [1].

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Chapter 1

Introduction

Over the past century, our view of physical phenomena has been revolutionized by two very different theories. On one hand, Quantum Mechanics accurately predicts the results of experiments taking place at small length scales. On the other hand, General Relativity correctly describes events at larger scales. However, while both theories have been impressively successful in their own ranges of application, our understanding of the interface between them is incomplete and their unification still remains one of the most prominent open problems in contemporary physics [2].

Although some proposals have been put forward for theories of unification, the lack of experiments has not allowed for the much-desired development. In fact, testing General Relativity at small lengths, where quantum mechanical effects become relevant, is highly non-trivial and has proven particularly difficult to achieve. The study of the opposite regime, however, has recently been growing in interest among researchers: it seems consensual that the necessary experiments to test Quantum Mechanics at large scales, where relativistic effects become important, will be within reach in the near future [3, 4, 5]. Although currently these experiments are primarily being designed to advance quantum technology, they will also allow for studying of the interplay between quantum theory and gravity.

In fact, in recent years, the rapid development of quantum technologies, such as those related to quantum communication [6, 7] and quantum metrology [8], has been one of the main propellers for the study of quantum theory [9, 10]. We are now entering an exciting new era of technology wherein, soon, we shall be able to exploit concrete applications of the physics of the quantum world, such as quantum cryptography [11, 12] and quantum teleportation [13], to achieve tasks not possible in classical contexts.

One of the primary goals of quantum technology is to develop a global network of quantum communication. Its applications could ultimately lead to the set up of a global quantum internet, which would translate into an exponential speed up in distributed computation, while, at the same time, being impenetrable to hackers [2]. However, to achieve this goal, quantum systems will have to operate over large distances, such as those from the Earth to orbiting satellites and, thus, relativistic and gravitational effects on quantum properties will have to be considered.

Quantum entanglement, the property of multipartite quantum systems in which measurement in one subsystem immediately affects the state of the other subsystems [9], is the primary feature of the quantum world upon which most of the new technologies are being built. Therefore, it is quite relevant to study the behaviour of entangled particles separated by large distances, as this will be crucial for the creation of global quantum communication schemes.

The first steps in this direction have already begun to be taken. Indeed, recently, a team of researchers has shown that observers separated by 144 km can share a quantum cryptographic key [14] by exploiting the randomness and strong correlations inherent to quantum entanglement. This experiment was performed by sharing a bipartite state of polarization-entangled photons via a free-space link between two telescopes in the Canary islands. Moreover, just last year, the same team was also able to successfully implement quantum teleportation across 143 km [15]. These two experiments set records for the distance achieved for tests of both quantum entanglement and quantum teleportation.

Following these developments, several proposals have been made to study quantum entanglement and teleportation in space-based missions using satellites [3, 4, 5]. For example, the group responsible for the experiments in the Canary islands, in collaboration with the Chinese Academy of Sciences, is also planning to demonstrate satellite-based quantum teleportation later this year [16]. An updated list of long-range experiments of quantum entanglement currently under way can be found in Ref. [2].

However, despite this progress, we still have a fairly limited theoretical background about how gravity and motion affect these quantum properties. So far, theoretical studies of quantum information have shown that quantum entanglement can be affected by non-uniform accelerations and, in theory, applying the equivalence principle, also by changing gravitational fields [17]. Also relevant is the fact that photons are red-shifted as they travel through the Earth's gravitational field, which could affect potential quantum communication protocols [5]. Thus, as experiments become more precise and the involved distances increase, a degradation of quantum coherence and entanglement is expected to become more important and needs to be considered.

Inspired by these experiments, we consider here the effect of Earth's gravitational field on entangled states of neutrons. The first experimental evidence for gravitationally bound quantum states of neutrons was reported in Ref. [18]. In the experiment, the particles are allowed to fall towards a horizontal mirror that, in conjunction with the Earth's gravitational field, serves as a confining potential well. Under these conditions, and as predicted by quantum theory, the falling neutrons acquire a discrete energy spectrum: rather than moving continuously along the vertical direction, the particles jump from a well-defined height to another. We use these states of the gravitational quantum well (GQW) [18, 19] and the theory of continuous variable entanglement [10, 20] to analyze the effect of having entangled particles at different values of the gravitational potential. We also discuss how this could affect an entanglement protocol between an observer in a station on the surface of the Earth and a second observer at a different height, possibly even at a satellite-based station in Low-Earth Orbit. Prior work on this topic can be found in Refs. [21, 22].

This dissertation is organized as follows: In Chapter 2 we present the theoretical background of Quantum Mechanics required for our analysis. In Chapter 3, we introduce the mathematical tools to study entanglement in a bipartite state of the GQW. In Chapter 4, we study the effects of considering two particles at different potentials. Finally, in Chapter 5, we discuss our results and present our conclusions.

Chapter 2

Theoretical Background

In this chapter, we introduce the theoretical tools required to analyze our system. We start by reviewing some aspects of the phase space formulation of Quantum Mechanics, with special emphasis on the Wigner quasi-probability distribution. Then, we introduce the subject of continuous variable information theory and discuss continuous variable tests of entanglement. Finally, we present the quantum-mechanical problem of a particle under the influence of a constant gravitational field and detail the experiment that verifies the GQW. Throughout this dissertation, we use units where $\hbar = 1$.

2.1 Quantum Mechanics in Phase Space

It is well known that a quantum mechanical state can be expressed in several equivalent representations. For example, the wave function in the position representation is the natural tool to express probability distributions in configuration space. On the other hand, if we are interested in visualizing the momentum distribution, the momentum representation of the wave function should be used. The phase space description is a intermediate between the position and the momentum representations and aims at treating these two variables in equal footing.

The Wigner quasi-probability distribution is one of the most widely used formulations of quantum-mechanical phase space. In this description, the density operators ρ of quantum systems are put into one-to-one correspondence with real-valued functions over a $2N$-dimensional phase space through the rule [10, 23]

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{\pi^N}\int_{\mathbb{R}^N} \langle \mathbf{x}-\mathbf{q}|\rho|\mathbf{x}+\mathbf{q}\rangle e^{2i\mathbf{q}\cdot\mathbf{p}} d^N\mathbf{q}, \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{p} = (p_1, p_2, ..., p_N)$, and the integrals run from $-\infty$ to $+\infty$ in all variables. It is clear that this map is invertible, and thus the function $W(\mathbf{x},\mathbf{p})$ captures the density matrix in its entirety.

One of the most useful properties of this description is related to marginal distributions: by performing a marginal integration of the Wigner function over $2N-1$ of its variables we obtain the correct probability distribution associated with the remaining quadrature [10]. For instance, integrating over all variables except for x_N yields the probability distribution for x_N:

$$\int_{\mathbb{R}^{2N-1}} W(\mathbf{x},\mathbf{p}) dp_1...dp_N dx_1...dx_{N-1} = \langle x_N|\rho|x_N\rangle. \tag{2.2}$$

The three basic properties of the density operator ρ can also be transcribed through the map (2.1) and expressed in the Wigner representation [23]. It can be shown that the hermiticity of the density operator, $\rho = \rho^\dagger$, is equivalent to the phase space distribution being real-valued, while the condition $\mathrm{tr}(\rho) = 1$ easily transcribes into the normalization $\int W(\mathbf{x},\mathbf{p}) d^N\mathbf{x} d^N\mathbf{p} = 1$.

These two conditions, along with the marginal distributions property, seem to give the Wigner function an intuitive interpretation as a joint probability distribution in position and momentum. However, the third condition for ρ, $\langle u|\rho|v\rangle \geq 0$, does not imply nonnegativity of $W(\mathbf{x},\mathbf{p})$. The Wigner function can, in fact, take both positive and negative values and, hence, does not satisfy all the axioms of probability theory. It should be regarded, instead, as a quasi-probability distribution.

It is also possible to show that any operator in Hilbert space can be mapped into a position and momentum function in phase space, via the Wigner transform [24]. Consider an operator $\hat{R}$ in Hilbert space. Then, its Wigner representation is obtained by

$$R(\mathbf{x}, \mathbf{p}) = \int_{\mathbb{R}^N} \langle \mathbf{x} - \mathbf{q}|\hat{R}|\mathbf{x} + \mathbf{q}\rangle e^{2i\mathbf{q}\cdot\mathbf{p}} d^N \mathbf{q}. \tag{2.3}$$

As a general rule, if there are no order ambiguities in quantization, the Wigner transforms correspond to mapping $\hat{x}_i \to x_i$ and $\hat{p}_i \to p_i$. However, if such ambiguities are present, Weyl ordering must be taken into account [25].

It is now easy to determine the average of the dynamical variable $\hat{R}$ in the state ρ using the Wigner function:

$$\langle \hat{R} \rangle = \mathrm{tr}(\rho \hat{R}) = \int_{\mathbb{R}^{2N}} W(\mathbf{x}, \mathbf{p}) R(\mathbf{x}, \mathbf{p}) dx_1 ... dx_N dp_1 ... dp_N. \tag{2.4}$$

The function $W(\mathbf{x}, \mathbf{p})$ acts, thus, as a weight probability distribution. The similarity of this formula to a classical phase space average is responsible for much of the intuitive appeal and practical utility of the Wigner representation [26].

2.2 Entanglement in Continuous Variable Systems

Quantum entanglement, the property of multipartite quantum systems in which measurement in one subsystem immediately affects the state of the other subsystems [9], has gained a new dimension of practical utility in recent years. Entanglement is nowadays regarded as a quantum resource that can be manipulated, controlled, and distributed. It plays a substantial role in the development of new quantum technologies, such as quantum cryptography [11, 12] and quantum teleportation [13].

The realization that quantum systems can be used as efficient tools for processing and transmitting information has led to the establishment of a new interdisciplinary domain, the research subject of Quantum Information Theory. It results from the effort to generalize classical information theory to the quantum world and focuses on studying how to harness the features of quantum systems [23].

Traditionally, two approaches to quantum information processing can be taken [10]. The first, often regarded as the "digital" approach, works with information encoded in systems with a discrete and finite number of degrees of freedom, such as nuclear spins or polarization of photons. The second approach, sometimes portrayed, in opposition, as "analog", works instead with correlations encoded in degrees of freedom with a continuous spectrum, such as the continuous variables associated with position and momentum of a particle.

Quantum protocols have traditionally been designed first from a discrete variable standpoint. This approach, due to the finite nature of the associated Hilbert spaces, is often mathematically simpler than that of continuous variable (CV) systems. However, in recent years, the field of CV quantum information has seen considerable progress, mainly due to the versatility provided by the large and rich structure of CV systems.

Characterization and detection of entanglement in CV systems is, nevertheless, a difficult task. The most successful strategy to deal with this difficulty has been to focus on the study of Gaussian states and operations [10], which form a resourceful basis for quantum information protocols and have been proven to be important testbeds for investigating quantum correlations.

Gaussian states are defined as those whose characteristic functions and quasi-probability distributions (namely their Wigner distributions) are Gaussian functions in the quantum phase space. These states are easy to produce and control with linear optical elements and, thus, appear ubiquitously in the laboratories of quantum physicists. Important examples include the vacuum, coherent, and thermal states of the electromagnetic field [20].

By definition, a Gaussian state can be completely described by the first and second statistical moments of the canonical operators. However, when addressing physical properties that are invariant under local unitary transformations, such as entanglement, we can neglect the first moments and completely characterize an N-mode Gaussian state by its corresponding $2N \times 2N$ covariance matrix $\boldsymbol{\sigma}$. The entries of $\boldsymbol{\sigma}$ correspond to the second order moments and are determined by the relations [10]

$$\sigma_{ij} = \langle \hat{X}_i \hat{X}_j + \hat{X}_j \hat{X}_i \rangle - 2\langle \hat{X}_i \rangle \langle \hat{X}_j \rangle, \tag{2.5}$$

where $\hat{\boldsymbol{X}} = (\hat{x}_1, \hat{p}_1, ..., \hat{x}_N, \hat{p}_N)^T$ is a vector of the quadrature operators, and $\langle \hat{O} \rangle$ denotes the mean value of the operator $\hat{O}$ evaluated at the corresponding state. The covariance matrix of a Gaussian state can, however, be identified simply by inspection of its corresponding Wigner function.

Note that although covariance matrices are of special importance to describe Gaussian states, they can also be built for any kind of non-Gaussian state as well. As we shall see, these matrices play an important role on the detection of entanglement in CV systems.

To study CV entanglement, we focus now on the particular case of two-mode states. These states serve as prototypical representations of bipartite CV quantum systems and, thus, establish an ideal test-ground for the investigation of quantum correlations [20]. Their covariance matrices can be written in the general block form

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C}^T & \boldsymbol{B} \end{pmatrix}, \tag{2.6}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are, respectively, the 2×2 covariance matrices of the two reduced modes A and B, and the correlation matrix between them.

A particularly useful test for checking entanglement or separability of a bipartite state in CV systems is given by the CV generalization of the Peres-Horodecki criterion, derived in Ref. [27]. For a general state described by the covariance matrix of Eq. (2.6), the criterion takes the form

$$\det\boldsymbol{A}\,\det\boldsymbol{B} + \left(\frac{1}{4} - |\det\boldsymbol{C}|\right)^2 - \mathrm{tr}(\boldsymbol{AJCJBJC}^T\boldsymbol{J}) \geq \frac{1}{4}(\det\boldsymbol{A} + \det\boldsymbol{B}), \tag{2.7}$$

where $\boldsymbol{J} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is an auxiliary matrix. Since this condition involves only moments up to second order, it is classified as a Gaussian test of entanglement.

It can be shown [27] that for two-mode Gaussian states, the inequality of Eq. (2.7) serves as a necessary and sufficient condition of separability. Thus, a verification of the inequality implies separability of the Gaussian state, while any violation immediately implies entanglement.

However, when this test is applied to a covariance matrix of a non-Gaussian state, it acts only as a necessary condition: any separable state must satisfy inequality (2.7). Thus, if the inequality is violated, we immediately conclude that the state must be entangled. On the other hand, if the condition is verified, no conclusion can be extracted.

This limitation is common to other second order tests and, hence, genuine entanglement of non-Gaussian states is often only revealed through application of criteria involving higher-order moments [28]. A particularly powerful separability criterion has been derived by Shchukin and Vogel that includes all the Gaussian criteria as special cases [29, 30]. The idea of this criterion is that, to any two-mode state ρ, we can assign a matrix of moments of the form

$$M_{ij} = \mathrm{tr}(\hat{a}^{\dagger q}\hat{a}^{p}\hat{a}^{\dagger n}\hat{a}^{m} \otimes \hat{b}^{\dagger l}\hat{b}^{k}\hat{b}^{\dagger r}\hat{b}^{s}\rho), \tag{2.8}$$

where $i = (pqrs)$ and $j = (nmkl)$. The operators $\hat{a}$ and $\hat{b}$ are built from the quadrature operators of, respectively, subsystems A and B [9]. It follows that the matrix of Eq. (2.8) is positive if and only if the state ρ remains positive under partial transposition, that is, if the state is separable. Thus, if any subdeterminant of the matrix M is negative, the state is revealed to be entangled. We shall use an extension of this criterion [31] to test the entanglement of our system in the next chapter of this work.

Finally, we direct the interested reader to Refs. [9, 23] for comprehensive discussions on the subject of quantum entanglement and, in particular, entanglement in CV systems.

2.3 The Gravitational Quantum Well

The problem of a particle subjected to a constant gravitational field is well known in quantum mechanics [32]. Consider a particle of mass M in a gravitational field $\mathbf{g} = -g\mathbf{e_x}$. When an horizontal mirror is placed at $x = 0$, the particle is constrained to the region $x \geq 0$ and a gravitational quantum well is established [18, 19].

This system is described by a linear potential of the form

$$V(x) = Mgx, \quad \text{for } x \geq 0. \tag{2.9}$$

Applying the potential energy to the Hamiltonian and solving the eigenvalue equation, $H\psi_n = E_n\psi_n$, it can be shown that the resulting eigenfunctions can be expressed in terms of the Airy function of first type,

$$\psi_n(x) = A_n \mathrm{Ai}\left(\frac{x - x_n}{x_0}\right), \tag{2.10}$$

while the corresponding energy eigenvalues are determined by the Airy function's roots, α_n, with $n = 1, 2, ...$:

$$E_n = -\left(\frac{Mg^2}{2}\right)^{1/3} \alpha_n. \tag{2.11}$$

The previous results introduce a handful of quantities that need to be identified. A_n is the normalization factor for the n-th level, which is determined by requiring $\int_0^\infty \psi_n^* \psi_n dx = 1$. This yields

$$A_n = \left[\int_0^\infty \mathrm{Ai}^2\left(\frac{x - x_n}{x_0}\right) dx\right]^{-1/2} = \frac{1}{x_0^{1/2}\mathrm{Ai}'(\alpha_n)}, \tag{2.12}$$

where $\mathrm{Ai}'(x)$ denotes the first derivative of the Airy function. The term x_0 is a scaling factor for the position variable and takes the value $x_0 \equiv (2M^2g)^{-1/3}$. Finally, $x_n = E_n/Mg = -x_0\alpha_n$ corresponds to the maximum height classically allowed for a particle with energy E_n.

The squared wavefunctions, ψ_n^2, for the first three energy levels are depicted

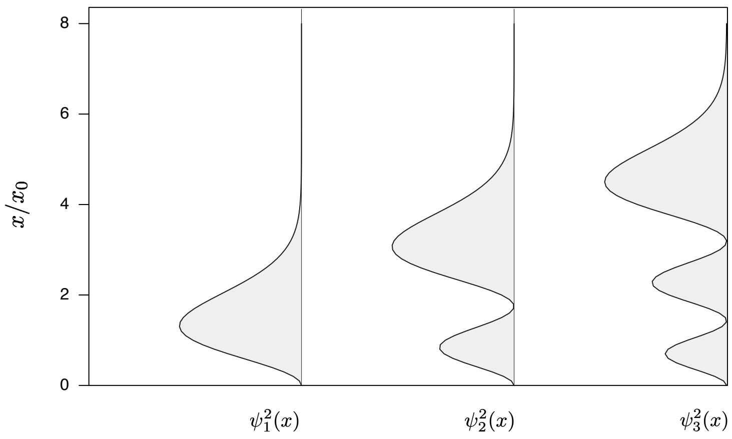

Figure 2.1: Squared wavefunctions for the first three quantum states of particles in the GQW.

in Figure (2.1). It is clear that the probability of finding the particle is non-vanishing for all values of $x > 0$. The wavefunctions display an oscillatory pattern until they reach a maximum value at the classical turning point $x = x_n$. Above this height, the probability of finding the particle decays exponentially.

This quantum system was built experimentally by submitting a beam of ultra-cold neutrons to Earth's gravitational well that bounces on a horizontal mirror [18]. In simple terms, the experiment runs as follows: a scatterer/absorber is placed above the horizontal mirror, forming a slit, and the neutron transmission through this slit is measured. If the height of the the scatterer/absorber is larger than the classical turning point for a given quantum state, the neutrons pass through the slit without loss. As the size of the slit decreases, the probability of neutron loss increases until the slit size reaches x_n and the apparatus stops being transparent to neutrons in the n-th quantum state. This procedure allows also for a criterion for the transition from quantum to classical behaviour [33].

Ultra-cold neutrons are fundamental in this experiment, as they are less likely to be affected by the electromagnetic interaction. Moreover, their relatively long lifetime and mass also allow for optimal conditions to establish the GQW.

Chapter 3

Entanglement in the Gravitational Quantum Well

We start by considering a bipartite state of the form

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} \left(|nm\rangle + |mn\rangle\right), \tag{3.1}$$

where $|n\rangle$ denotes the n-th level of the GQW, that is, the state of a neutron in a gravitational field $\mathbf{g}$ with energy E_n and wavefunction $\langle x|n\rangle = \psi_n(x)$. When studying bipartite systems we use $|nm\rangle = |n\rangle_A \otimes |m\rangle_B$ to denote a two-particle system with a particle in the energy level n in subsystem A and a particle in the energy level m in subsystem B.

To study CV entanglement in this system, we proceed by determining its corresponding Wigner description in phase space. This will allow us to more easily calculate the statistical moments in the position and momentum variables required to perform tests of entanglement.

First, we consider the density matrix for the state of Eq. (3.1):

$$\rho = |\psi^+\rangle\langle\psi^+| = \frac{1}{2} \left(|nm\rangle\langle nm| + |nm\rangle\langle mn| + |mn\rangle\langle nm| + |mn\rangle\langle mn|\right). \tag{3.2}$$

We now proceed by applying this density matrix to the definition of the Wigner function. Since we are working with a system of two particles, we set $N = 2$

in Eq. (2.1) and the definition takes the form

$$W(x_A, x_B; p_A, p_B) = \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \langle x_A - q_A, x_B - q_B | \rho | x_A + q_A, x_B + q_B \rangle \\ \times e^{2i(p_A q_A + p_B q_B)} dq_A dq_B. \tag{3.3}$$

Next, we notice that $\langle x - q | n \rangle \langle n | x + q \rangle = A_n^2 \mathrm{Ai}\left(\frac{x-q-x_n}{x_0}\right) \mathrm{Ai}\left(\frac{x+q-x_n}{x_0}\right)$. Thus, the computation of the Wigner function requires the use of the following results for integrals involving Airy functions [32]:

$$\int_{-\infty}^{+\infty} \mathrm{Ai}\left(\frac{x - x_n - q}{x_0}\right) \mathrm{Ai}\left(\frac{x - x_n + q}{x_0}\right) e^{2ipq} dq = \\ = \frac{x_0}{2^{1/3}} \mathrm{Ai}\left(2^{2/3}\left(\frac{x - x_n}{x_0} + x_0^2 p^2\right)\right), \tag{3.4}$$

$$\int_{-\infty}^{+\infty} \mathrm{Ai}\left(\frac{x - x_m - q}{x_0}\right) \mathrm{Ai}\left(\frac{x - x_n + q}{x_0}\right) e^{2ipq} dq = \\ = \frac{x_0}{2^{1/3}} \mathrm{Ai}\left(\frac{2x - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p^2\right) e^{i(x_n - x_m)p}. \tag{3.5}$$

Using these relationships, it is straightforward to show that the Wigner function for the bipartite state of Eq. (3.1) takes the form:

$$\begin{aligned} &W(x_A, x_B; p_A, p_B) = \\ &\frac{A_n^2 A_m^2}{2\pi^2} \frac{x_0^2}{2^{2/3}} \Bigg\{ \mathrm{Ai}\left(2^{2/3}\left(\frac{x_A - x_n}{x_0} + x_0^2 p_A^2\right)\right) \mathrm{Ai}\left(2^{2/3}\left(\frac{x_B - x_m}{x_0} + x_0^2 p_B^2\right)\right) \\ &+ \mathrm{Ai}\left(\frac{2x_A - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p_A^2\right) e^{i(x_n - x_m)p_A} \\ &\qquad \times \mathrm{Ai}\left(\frac{2x_B - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p_B^2\right) e^{i(x_m - x_n)p_B} \\ &+ \mathrm{Ai}\left(\frac{2x_A - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p_A^2\right) e^{i(x_m - x_n)p_A} \\ &\qquad \times \mathrm{Ai}\left(\frac{2x_B - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p_B^2\right) e^{i(x_n - x_m)p_B} \\ &+ \mathrm{Ai}\left(2^{2/3}\left(\frac{x_A - x_m}{x_0} + x_0^2 p_A^2\right)\right) \mathrm{Ai}\left(2^{2/3}\left(\frac{x_B - x_n}{x_0} + x_0^2 p_B^2\right)\right) \Bigg\}. \end{aligned} \tag{3.6}$$

It is clear that the system we are considering is not Gaussian and, thus, it is not possible to identify its corresponding covariance matrix simply by inspection of the Wigner function. However, having a phase-space description of the system, we are in condition to extract the statistical moments of the state and build the necessary matrices of moments to test CV entanglement. We shall start by building the covariance matrix of this state and proceed by applying the most basic test of CV entanglement, the generalization of the Peres-Horodecki criterion.

As we have seen in the previous chapter, the n-th statistical moment of an operator $\hat{O}$ can be extracted from the Wigner function:

$$\langle O^n \rangle = \int W(x_A, x_B; p_A, p_B) O^n(x_A, x_B; p_A, p_B) dx_A dx_B dp_A dp_B, \tag{3.7}$$

where the integrals are calculated over the allowed region of phase space. For the GQW, this corresponds to the range $x_i \in [0, +\infty[$ in the position variables and $p_i \in \mathbb{R}$ in the momentum variables (i = A, B).

We are interested in determining moments up to second order in position and momentum. To achieve this, we see from Eq. (3.6) that we have to calculate two types of integrals. We call integrals of type I those of the form

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} O(x,p) \mathrm{Ai}\left(2^{2/3}\left(\frac{x - x_n}{x_0} + x_0^2 p^2\right)\right) dp dx, \tag{3.8}$$

and type II those of the form

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} O(x,p) \mathrm{Ai}\left(\frac{2x - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p^2\right) e^{i(x_n - x_m)p} dp dx, \tag{3.9}$$

where $O(x,p)$ represents the combination of variables corresponding to the expected statistical moment.

Most of these integrals are not standard, and thus require a considerable amount of manipulations to be computed. Those of the form of Eq. (3.9) are particularly challenging to evaluate. However, Airy functions possess many algebraic and cyclic properties that can be exploited in order to calculate these integrals. Ref. [32] is an excellent resource for this task.

Performing these calculations, we arrive at the results presented at Table (3.1).

$O(x,p)$	Type I	Type II
1	$\frac{2^{1/3}\pi}{x_0 A_n^2}$	0
x	$-\frac{2^{4/3}\pi}{3A_n^2}\alpha_n$	$-\frac{2^{4/3}\pi}{A_n A_m}\left(\frac{1}{\alpha_m-\alpha_n}\right)^2$
x^2	$\frac{2^{1/3}8\pi x_0}{15A_n^2}\alpha_n^2$	$-\frac{2^{1/3}24\pi x_0}{A_n A_m}\left(\frac{1}{\alpha_m-\alpha_n}\right)^4$
p	0	$-\frac{2\pi}{x_0^2 A_n A_m}\left(\frac{1}{\alpha_m-\alpha_n}\right)$
p^2	$-\frac{2^{4/3}\pi}{3x_0^3 A_n^2}\alpha_n^2$	$-\frac{2^{2/3}4\pi}{x_0^3 A_n A_m}\left(\frac{1}{\alpha_m-\alpha_n}\right)^2$
xp	0	$\frac{12\pi}{x_0 A_n A_m}\left(\frac{1}{\alpha_m-\alpha_n}\right)^3$

Table 3.1: Results for the integrals of Eqs. (3.8) and (3.9), required for the calculation of statistical moments, for various combinations $O(x,p)$ of position and momentum variables.

From here, we build the moments by matching the results of Table (3.1) in the way required by the Wigner function. From Eq. (3.6), we see that the function is composed of four terms, each of which requires the calculation of two integrals, one for subsystem A and another for subsystem B. Two of these terms involve only integrals of type I, while the other two involve solely integrals of type II.

As an example of calculation, we use the results for $O(x,p) = 1$ to verify that

$$\begin{aligned}\int W(x_A, x_B; p_A, p_B) dx_A dx_B dp_A dp_B &= \frac{A_n^2 A_m^2}{2\pi^2}\frac{x_0^2}{2^{2/3}}\left\{\frac{2^{1/3}\pi}{x_0 A_n^2}\frac{2^{1/3}\pi}{x_0 A_m^2} + \frac{2^{1/3}\pi}{x_0 A_m^2}\frac{2^{1/3}\pi}{x_0 A_n^2}\right\} \\ &= 1,\end{aligned} \tag{3.10}$$

which, as we have seen, is one of the main properties of any Wigner function. The only terms contributing to this calculation are those of integrals of type I, since those of type II vanish.

For a more detailed example of calculation, we direct the reader to the Appendix, where the of computation of $\langle x_A \rangle$ is presented thoroughly, including the calculation of all the relevant integrals for that case.

We now use Eq. (2.5) to build the covariance matrix of the bipartite state, which can be shown to take the following form:

$$\boldsymbol{\sigma} = \begin{pmatrix} x_0^2\alpha & 0 & x_0^2\beta & 0 \\ 0 & \gamma/x_0^2 & 0 & \delta/x_0^2 \\ x_0^2\beta & 0 & x_0^2\alpha & 0 \\ 0 & \delta/x_0^2 & 0 & \gamma/x_0^2 \end{pmatrix}, \tag{3.11}$$

where

$$\alpha = \frac{14}{45}\left(\alpha_n^2 + \alpha_m^2\right) - \frac{4}{9}\alpha_n\alpha_m \tag{3.12}$$

$$\beta = \frac{8}{\left(\alpha_m - \alpha_n\right)^2} - \frac{2}{9}\left(\alpha_m - \alpha_n\right)^2 \tag{3.13}$$

$$\gamma = -4\left(\alpha_m + \alpha_n\right) \tag{3.14}$$

$$\delta = -2\left(\frac{2^{2/3}}{\alpha_m - \alpha_n}\right)^2 \tag{3.15}$$

As it has already been referred, for the study Gaussian entanglement we would simply have to apply a Gaussian test of entanglement to this matrix. A necessary and sufficient condition for separability of Gaussian states is given by the Peres-Horodecki criterion [27], which reduces to the inequality of Eq. (2.7). When applied to matrix of Eq. (3.11), the criterion takes the form $\Delta \geq 0$, where

$$\Delta = (\alpha\gamma)^2 - (\alpha\delta)^2 - (\beta\gamma)^2 + \frac{1}{16} - \frac{|\beta\delta|}{2} + (\beta\delta)^2 - \frac{\alpha\gamma}{2}. \tag{3.16}$$

It is easy to show, using the numerical values of the zeros α_n, that all GQW states satisfy this condition and, thus, there is no evidence of entanglement at this level. In fact, considering the two lowest energy levels, $n = 1$ and $m = 2$, the criterion yields $\Delta = 4573.31 > 0$. For higher values of n and m, the value in the left-hand side becomes larger and the separability criterion is

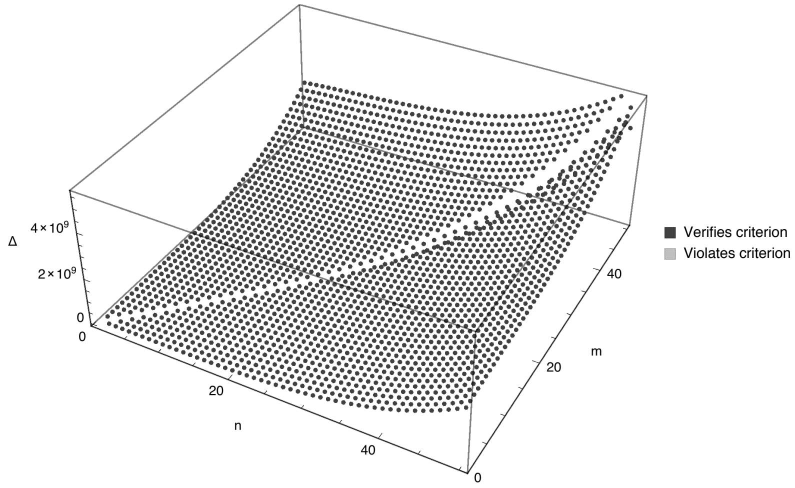

Figure 3.1: Numerical results for the application of the Peres-Horodecki criterion for various combinations of n and m, up to the energy level 50.

never broken. Fig. (3.1) shows the numerical results for the application of the criterion for different energy levels.

However, the CV generalization of the Peres-Horodecki criterion is only a necessary and sufficient condition of separability when applied to Gaussian states. For non-Gaussian states, second order criteria may fail to reveal entanglement [28]. Thus, when Gaussian tests fail, genuine entanglement of non-Gaussian states may only revealed through application of criteria involving higher-order moments.

To achieve this, we follow the construction presented in Ref. [31], where it is developed a generalization of the Positive Partial Transpose (PPT) criterion for CV systems based on the matrices of moments. The PPT criterion states that a separable state remains positive under partial transposition, and, therefore, a Non-positive-Partial-Transposition (NPT) state must be entangled. As we have seen in section (2.2), the generalization to CV systems depends on the fact that a matrix of moments of the form of Eq. (2.8) is positive if and only if the corresponding state is PPT, that is, if the state is separable [9].

We start by defining adimensional operators a and $a^\dagger$ for subsystem A, and

b and $b^\dagger$ for subsystem B, such that

$$a = \frac{1}{\sqrt{2}}\left(\frac{x_A}{x_0} + ix_0 p_A\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x_A}{x_0} - ix_0 p_A\right), \tag{3.17}$$

$$b = \frac{1}{\sqrt{2}}\left(\frac{x_B}{x_0} + ix_0 p_B\right), \quad b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x_B}{x_0} - ix_0 p_B\right). \tag{3.18}$$

We build now a suitable matrix of moments $M_f(\rho) = [M_{ij}] = [\langle f_i^\dagger f_j \rangle]$ that forms the basis for the criterion. Let ρ^Γ denote partial transposition of the state ρ with respect to subsystem B. Then, the criterion reads as follows: *a bipartite state ρ is NPT if and only if there exists f such that* $\det M_f(\rho^\Gamma)$ *is negative* [31]. Therefore, to reveal entanglement in the system, we seek a class of operators f whose corresponding matrix of moments yields a negative determinant.

Moreover, it can be shown that if the class of operators f has a tensor product structure, $\tilde{f} = f^A \otimes f^B$, then the matrix of moments of the partially-transposed state is equal to the partial transpose of the matrix of moments of the original state, $M_{\tilde{f}}(\rho^\Gamma) = (M_{\tilde{f}}(\rho))^\Gamma$. To take advantage of this, we choose $\tilde{f} = (1, a) \otimes (1, b) = (1, a, b, ab)$ and the corresponding matrix of moments becomes

$$M_{\tilde{f}}(\rho) = \begin{pmatrix} 1 & \langle a \rangle & \langle b \rangle & \langle ab \rangle \\ \langle a^\dagger \rangle & \langle a^\dagger a \rangle & \langle a^\dagger b \rangle & \langle a^\dagger ab \rangle \\ \langle b^\dagger \rangle & \langle ab^\dagger \rangle & \langle b^\dagger b \rangle & \langle ab^\dagger b \rangle \\ \langle a^\dagger b^\dagger \rangle & \langle a^\dagger ab^\dagger \rangle & \langle a^\dagger b^\dagger b \rangle & \langle a^\dagger ab^\dagger b \rangle \end{pmatrix}. \tag{3.19}$$

The next step consists in rewriting the statistical moments of the matrix of Eq. (3.19) in terms of the statistical moments of the momentum and position variables, taking into account the definitions of Eqs. (3.17) and (3.18). Performing this allows us to determine the entries of $M_{\tilde{f}}(\rho)$ by applying the results from Table (3.1). Then, we apply the rules of partial transposition to obtain $(M_{\tilde{f}}(\rho))^\Gamma$.

Substituting the numerical values for the zeros of Airy functions α_n, it can be shown that any combination of GQW states n and m satisfies the condition

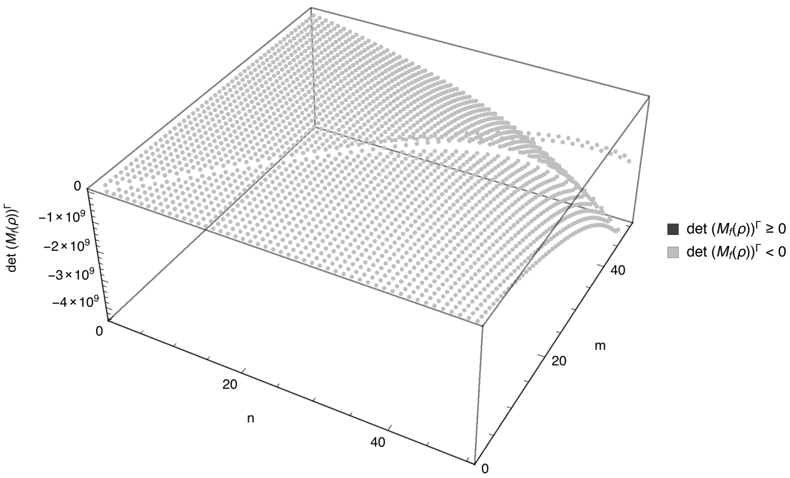

Figure 3.2: Numerical results for the application of the non-Gaussian generalization of the PPT criterion for various combinations of n and m, up to the energy level 50.

$\det(M_{\tilde{f}}(\rho))^\Gamma < 0$. These results are plotted in Fig. (3.2). As an example, if we consider $n = 1$ and $m = 2$, $\det(M_{\tilde{f}}(\rho))^\Gamma = -0.588169$. For higher energy levels, this value remains negative. Hence, we conclude that the state described by Eq. (3.1) is NPT and the system is entangled.

Chapter 4

Moving in the Gravitational Field

We aim now to study the behavior of this entangled system when one of its parts moves vertically in the gravitational field. To achieve this, we consider that the particle in subsystem B is displaced to a height H relative to that of subsystem A and, at this new position, the particle feels a gravitational field in the radial direction with strength $g' < g$.

Looking back at the definitions of the GQW in Eqs. (2.10) and (2.11), we see that the new subsystem B is dependent on the constants $x_0' = (2M^2 g')^{-1/3}$ and $x_n' = -x_0' \alpha_n$, and the new energy levels are given by

$$E_n' = -\left(\frac{Mg'^2}{2}\right)^{1/3} \alpha_n = -Mg' x_0' \alpha_n. \tag{4.1}$$

We consider again a state of the form

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} \left(|nm'\rangle + |mn'\rangle\right), \tag{4.2}$$

where $|n'\rangle$ is the n-th energy level of the particle in potential g'.

The new Wigner function is, thus, dependent on both x_0 and x_0', respectively due to the positions of particle A and particle B. It is also important to note that new normalization factors A_n' must be included in the Wigner description. However, it is straightforward to see from the results of Table (3.1) that these normalization factors are easily factored out when we compute any type of statistical moment.

The statistical moments are obtained in the same fashion as those of the previous section: we can, again, distinguish between the two types of integrals and we build the moments by pairing the results of Table (3.1) in the way required by the Wigner function. The only difference is that now all the results involving subsystem B are affected by x'_0 instead of x_0.

We want to study how the criterion presented in the previous section is affected by this change in the gravitational field. For this purpose we build the adimensional operators of Eqs. (3.17) and (3.18) with the new statistical moments in B. Since these moments on the position and momentum variables now depend on x'_0, all moments in b and $b^\dagger$ become affected by the ratio

$$\frac{x'_0}{x_0} = \frac{(2M^2 g')^{-1/3}}{(2M^2 g)^{-1/3}} = \left(\frac{g}{g'}\right)^{1/3}. \tag{4.3}$$

We now build the matrix of moments of Eq. (3.19) and apply the CV generalization of the PPT criterion for different values of g/g'. The results are shown in Figures (4.1)-(4.4).

It is clear from the results of Figures (4.1) and (4.2) that if the particle B feels a gravitational field g' weaker than g, then the determinant of the matrix of moments remains negative and, thus, the system remains NPT. We expect, therefore, that moving the particle upwards in the gravitational field does not break the entanglement that we have examined in the previous chapter for the case where the two particles are at the same height.

Notice that if we could achieve a scenario where $g' > g$, the results would not be as simple. As we can see from Figures (4.3) and (4.4), for extreme cases of $g/g' < 1$ the determinant becomes positive and the generalization of PPT criterion fails to reveal entanglement. However, due to the nature of the criterion, it is unclear if the entanglement is indeed broken by the stronger gravitational field or if it is simply a limitation imposed by our choice for the

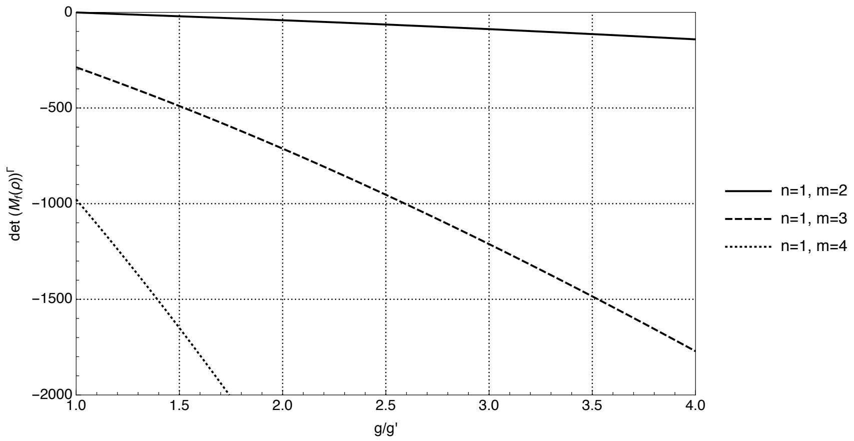

Figure 4.1: Numerical results for the application of the non-Gaussian generalization of the PPT criterion for different values of $g/g' > 1$ on three combinations of n and m

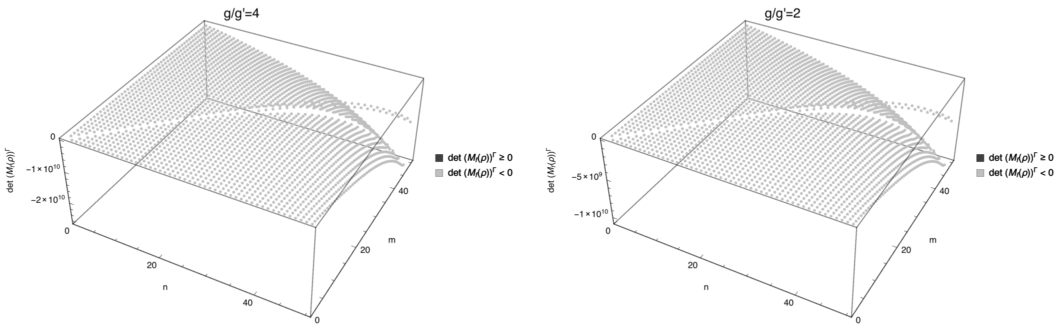

Figure 4.2: Numerical results for the application of the non-Gaussian generalization of the PPT criterion for two different values of the ratio $g/g' > 1$ and multiple combinations of n and m, up to the energy level 50.

class of operators $\tilde{f}$.

Following this, it is clear that if we could devise a system similar to that of Ref. [14], which was used to verify the presence of entanglement between two observers separated by 144 km, but with platforms at different heights, the entanglement of the states would not be disrupted by gravity.

We can think of a system composed of two parts: station A, located at the surface of the Earth; and station B, a platform at height H. Station A would be responsible for producing the entangled bipartite state and sending one of the

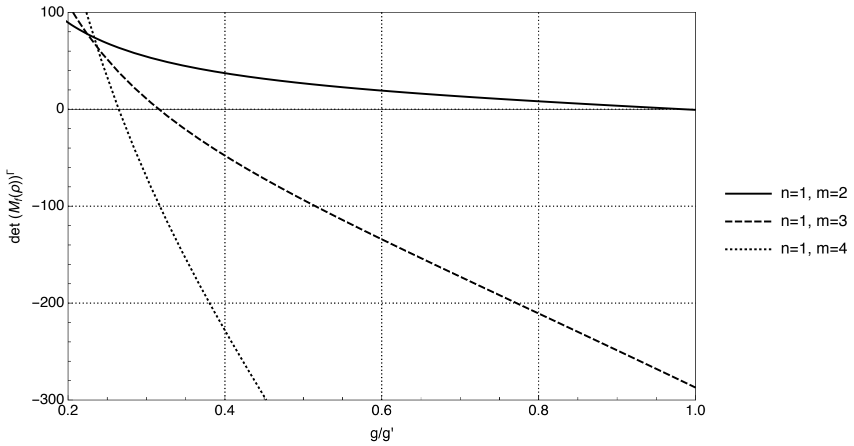

Figure 4.3: Numerical results for the application of the non-Gaussian generalization of the PPT criterion for different values of $g/g' < 1$ on three combinations of n and m

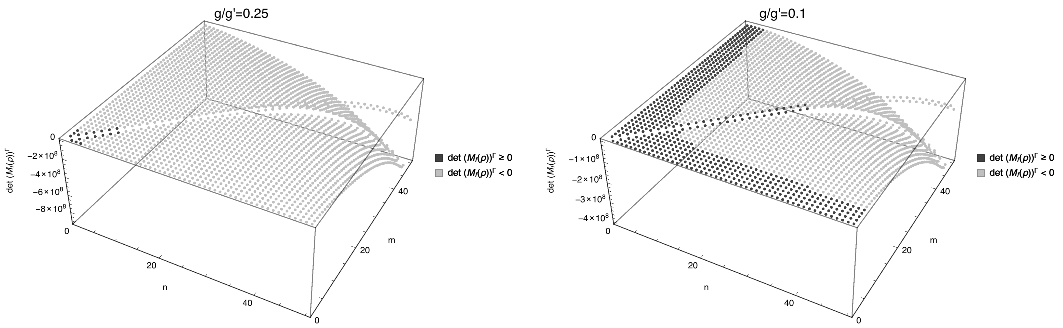

Figure 4.4: Numerical results for the application of the non-Gaussian generalization of the PPT criterion for two different values of the ratio $g/g' < 1$ and multiple combinations of n and m, up to the energy level 50.

particles to the other station. Ideally, station B would be an orbiting platform placed in Low-Earth Orbit (LEO), allowing for a separation of hundreds of kilometers between the two stations, that is, of the same order of magnitude as those of the experiments performed in the Canary islands [14, 15]. As we have seen, moving one of the particles to station B, which is in a weaker region of the gravitational field, should not break the entanglement of the bipartite system.

More details on the proposals for satellite-based tests of quantum entanglement and, in particular, for platforms at LEO can be found in Refs. [3, 4, 5].

Chapter 5

Conclusions

In this work we have studied the effect of gravity on the entanglement of states. We have built a phase-space description of a bipartite CV system of two particles in the gravitational quantum well by calculating its corresponding Wigner function. We have shown that this Wigner description corresponds to a non-Gaussian state.

We have approached the detection of entanglement in this CV system in a progressive fashion. First, we applied one of the most standard separability tests, the CV generalization of the Peres-Horodecki criterion, based on second-order statistical moments of the position and momentum variables. We concluded that there was no evidence of entanglement at that level and, thus, proceeded to tests involving higher-order moments. Using a more general criterion based on Positive Partial Transposition using matrices of moments, we were able to demonstrate the presence of entanglement for any combination of GQW energy levels.

Finally, we have examined the effects of considering particles at different gravitational potentials and shown that the entanglement of states persists even if one of the parts of the system moves to a weaker gravitational field. This result corroborates the feasibility of proposed satellite-based applications of quantum entanglement.

However, a few considerations must be made. The first is that the GQW can be easily destabilized, due to the weakness of the gravitational force. This is one of main reasons to use neutrons as entangling particles, as they are less likely to be affected by the electromagnetic interaction. Another is that, as we increase in height, we need to be more careful with the constant-field approximation on which the GQW results are based.

Having the latter constraint in mind, we have also studied the effects of considering higher-order terms in the expansion of the gravitational potential via perturbation theory. Considering the perturbation of a quadratic term in the distance, we have concluded that the results discussed are not invalidated. In fact, every correction term is affected by a factor x_0/R, where x_0 is a characteristic length of the GQW of the order of micrometers and R is the radius of the Earth. This ratio is so small that all corrections become irrelevant at this order.

Moreover, we have also tried to explore this system using the full inverse-distance gravitational potential. However, the lack of development regarding the Wigner phase-space formulation for the $1/x$ potential presented itself as a major setback in our investigation and we were not able to reach any results.

From here, a few lines of further work can be traced. First, it would be interesting to study the application of other (possibly more powerful) non-Gaussian entanglement criteria to this system, so as to clarify the behavior in the $g/g' < 1$ scenario. Another possibility, although this would require advancements in the field of CV information theory for non-Gaussian states, would be to apply measures of entanglement to the system instead of simply using markers, as we have done. These measures would allow for a quantification of the amount of entanglement present in the system, which would serve as a further tool to study degradation by the gravitational field. Finally, noncommutative extensions of this work could be taken, which would allow for further exploration of the interplay between noncommutativity of space-time and quantum entanglement.

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Appendix

The goal of this Appendix is to provide an example of some of the techniques required to compute the integrals of Eqs. (3.8) and (3.9), which give origin to Table (3.1) and were discussed in Chapter 3. It serves as well to clarify some aspects of the calculation of statistical moments using the four-part Wigner function of Eq. (3.6). We shall achieve this by exemplifying the calculation of $\langle x_A \rangle$ and all the integrals required for this case.

The calculation of integrals involving Airy functions is made easier by the cyclic and algebraic properties that this function possesses [32]. Two of the most useful properties are the cyclic relation imposed by the differential equation that the Airy function solves,

$$\mathrm{Ai}''(x) = x\mathrm{Ai}(x), \tag{A.1}$$

where $\mathrm{Ai}''(x)$ denotes the second derivative of the Airy function, and the following integral formulation of the Airy function,

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(z^3/3+xz)}dz. \tag{A.2}$$

We shall not use these properties explicitly in the examples we are presenting, since tabulated results of integrals involving Airy functions will suffice. However, relationships (A.1) and (A.2) are used implicitly to evaluate these standard integrals.

Following the discussion presented in Chapter 3, we see that the computation of $\langle x_A \rangle$ requires the integrals of Eqs. (3.8) and (3.9) both for $O(x, p) = 1$

and $O(x,p) = x$. Let us start with the calculation of the integral of type I, that is, with the form of Eq. (3.8), with $O(x,p) = 1$:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{0}^{+\infty} \mathrm{Ai}\left(2^{2/3}\left(\frac{x-x_n}{x_0}+x_0^2p^2\right)\right) dx\,dp \\
&\quad = 2\int_{0}^{+\infty}\int_{0}^{+\infty} \mathrm{Ai}\left(2^{2/3}\left(\frac{x-x_n}{x_0}+x_0^2p^2\right)\right) dx\,dp && \text{(A.3a)}\\
&\quad = \frac{2^{2/3}}{x_0}\int_{0}^{+\infty}\int_{0}^{+\infty} \mathrm{Ai}\left(2^{2/3}\left(\frac{x-x_n}{x_0}\right)+\tilde{p}^2\right) dx\,d\tilde{p} && \text{(A.3b)}\\
&\quad = \frac{2^{2/3}}{x_0}\int_{0}^{+\infty}\int_{0}^{+\infty} \mathrm{Ai}\left(2^{2/3}\left(\frac{x-x_n}{x_0}\right)+\xi\right) dx\,\frac{d\xi}{2\sqrt{\xi}} && \text{(A.3c)}\\
&\quad = \frac{2^{1/3}\pi}{x_0}\int_{0}^{+\infty} \mathrm{Ai}^2\left(\frac{x-x_n}{x_0}\right) dx && \text{(A.3d)}\\
&\quad = 2^{1/3}\pi\int_{0}^{+\infty} \mathrm{Ai}^2\left(\tilde{x}+\alpha_n\right) d\tilde{x} && \text{(A.3e)}\\
&\quad = 2^{1/3}\pi\int_{\alpha_n}^{+\infty} \mathrm{Ai}^2\left(\bar{x}\right) d\bar{x} && \text{(A.3f)}\\
&\quad = 2^{1/3}\pi\left\{-\alpha_n\mathrm{Ai}^2(\alpha_n)+\mathrm{Ai}'^{\,2}(\alpha_n)\right\} && \text{(A.3g)}\\
&\quad = \frac{2^{1/3}\pi}{x_0A_n^2}. && \text{(A.3h)}
\end{aligned}$$

In the last step, we have used the definition of the normalization A_n found in Eq. (2.12), as well as the fact that the function vanishes when evaluated on one of its roots, $\mathrm{Ai}(\alpha_n) = 0$. The calculation of the integral in the variable ξ in Eq. (A.3c) required the use of a known result for Airy functions found in Ref. [32]. This reference includes an extensive list of known integrals involving Airy functions, along with many of the function's most useful properties.

The integral of type I with $O(x,p) = x$ follows closely that with $O(x,p) = 1$. First, we solve the integral in p following similar steps to those of Eqs. (A.3a)-

(A.3d). Then, we use tabulated results for the remaining integrals in x:

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty} x\mathrm{Ai}\left(2^{2/3}\left(\frac{x-x_n}{x_0}+x_0^2p^2\right)\right)dx\,dp$$

$$= \frac{2^{1/3}\pi}{x_0}\int_{0}^{+\infty} x\mathrm{Ai}^2\left(\frac{x-x_n}{x_0}\right)dx \tag{A.4a}$$

$$= 2^{1/3}\pi x_0\int_{0}^{+\infty} \tilde{x}\mathrm{Ai}^2\left(\tilde{x}+\alpha_n\right)d\tilde{x} \tag{A.4b}$$

$$= 2^{1/3}\pi x_0\int_{\alpha_n}^{+\infty} (\bar{x}-\alpha_n)\mathrm{Ai}^2\left(\bar{x}\right)d\bar{x} \tag{A.4c}$$

$$= \frac{2^{1/3}\pi x_0}{3}\alpha_n\mathrm{Ai}'^2(\alpha_n) - 2^{1/3}\pi x_0\alpha_n\mathrm{Ai}'^2(\alpha_n) \tag{A.4d}$$

$$= -\frac{2^{4/3}}{3}\pi x_0\alpha_n\mathrm{Ai}'^2(\alpha_n) \tag{A.4e}$$

$$= -\frac{2^{4/3}\pi}{3A_n^2}\alpha_n, \tag{A.4f}$$

where, once again, we have used the fact that the function vanishes when evaluated on its roots, $\mathrm{Ai}(\alpha_n) = 0$, as well as the properties imposed by the limits [32]

$$\lim_{x\to\infty}\mathrm{Ai}(x+a) = \lim_{x\to\infty} x\,\mathrm{Ai}(x+a) = \lim_{x\to\infty}\mathrm{Ai}'(x+a) = 0. \tag{A.5}$$

Next, we proceed with the calculation of the integrals of type II, that is, those of the form of Eq. (3.9). This class of integrals is slightly more complex to compute, mainly due to the presence of the exponential dependence in p. However, they become quite manageable to execute if we use the relationships [32]

$$\int_{-\infty}^{+\infty}\mathrm{Ai}(x^2+a)e^{ikx}dx = 2^{2/3}\pi\mathrm{Ai}\left(2^{-2/3}(a-k)\right)\mathrm{Ai}\left(2^{-2/3}(a+k)\right), \tag{A.6}$$

and

$$\int_{0}^{+\infty} \mathrm{Ai}\left(\frac{x-x_m}{x_0}\right)\mathrm{Ai}\left(\frac{x-x_n}{x_0}\right) = 0, \tag{A.7}$$

where the result of Eq. (A.7) follows from the application of the limits in Eq. (A.5).

Then, the case $O(x,p) = 1$ reduces to

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{0}^{+\infty} \mathrm{Ai}\left(2^{-1/3}\left(\frac{2x-x_m-x_n}{x_0}\right)+2^{2/3}x_0^2p^2\right)e^{i(x_m-x_n)p}dx\,dp \\
&= \frac{1}{2^{1/3}x_0}\int_{0}^{+\infty}dx\int_{-\infty}^{+\infty}\mathrm{Ai}\left(2^{-1/3}\left(\frac{2x-x_m-x_n}{x_0}\right)+\tilde{p}^2\right)e^{i(x_m-x_n)\frac{\tilde{p}}{2^{1/3}x_0}}d\tilde{p} && \text{(A.8a)} \\
&= \frac{2^{1/3}\pi}{x_0}\int_{0}^{+\infty}\mathrm{Ai}\left(\frac{x-x_m}{x_0}\right)\mathrm{Ai}\left(\frac{x-x_n}{x_0}\right)dx && \text{(A.8b)} \\
&= 0. && \text{(A.8c)}
\end{aligned}$$

Similarly, for the case $O(x,p) = x$,

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{0}^{+\infty} x\mathrm{Ai}\left(2^{-1/3}\left(\frac{2x-x_m-x_n}{x_0}\right)+2^{2/3}x_0^2p^2\right)e^{i(x_m-x_n)p}dx\,dp \\
&= \frac{2^{1/3}\pi}{x_0}\int_{0}^{+\infty}x\mathrm{Ai}\left(\frac{x-x_m}{x_0}\right)\mathrm{Ai}\left(\frac{x-x_n}{x_0}\right)dx && \text{(A.9a)} \\
&= 2^{1/3}\pi x_0\int_{0}^{+\infty}\tilde{x}\mathrm{Ai}\left(\tilde{x}+\alpha_m\right)\mathrm{Ai}\left(\tilde{x}+\alpha_n\right)d\tilde{x} && \text{(A.9b)} \\
&= -\frac{2^{4/3}\pi x_0}{(\alpha_m-\alpha_n)^2}\mathrm{Ai}'(\alpha_m)\mathrm{Ai}'(\alpha_n) && \text{(A.9c)} \\
&= -\frac{2^{4/3}\pi}{A_mA_n}\left(\frac{1}{\alpha_m-\alpha_n}\right)^2, && \text{(A.9d)}
\end{aligned}$$

where Eq. (A.9b) is evaluated using a standard integral found in Ref. [32] in combination with the application of the limits in Eq. (A.5).

Having these results, we are now in conditions to evaluate $\langle x_A \rangle$:

$$\langle x_A \rangle = \int x_A \, W(x_A, x_B; p_A, p_B) dx_A \, dx_B \, dp_A \, dp_B. \tag{A.10}$$

The Wigner function of Eq. (3.6) consists of four terms, each of which containing an Airy function with dependence in x_A and p_A, and another Airy function with dependence in x_B and p_B. It follows that, to obtain $\langle x_A \rangle$, the integrals with dependence in the variables of subsystem A will correspond to those with $O(x, p) = x$, while the integrals with dependence in the variables of subsystem B will correspond to those with $O(x, p) = 1$:

$$\begin{aligned}
\langle x_A \rangle = \frac{A_n^2 A_m^2}{2\pi^2} \frac{x_0^2}{2^{2/3}} \Bigg\{ & \\
& \iint x_A \mathrm{Ai}\left(2^{2/3}\left(\frac{x_A - x_n}{x_0} + x_0^2 p_A^2\right)\right) dx_A \, dp_A \\
& \quad \times \iint \mathrm{Ai}\left(2^{2/3}\left(\frac{x_B - x_m}{x_0} + x_0^2 p_B^2\right)\right) dx_B \, dp_B \\
& + \iint x_A \mathrm{Ai}\left(\frac{2x_A - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p_A^2\right) e^{i(x_n - x_m)p_A} dx_A \, dp_A \\
& \quad \times \iint \mathrm{Ai}\left(\frac{2x_B - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p_B^2\right) e^{i(x_m - x_n)p_B} dx_B \, dp_B \\
& + \iint x_A \mathrm{Ai}\left(\frac{2x_A - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p_A^2\right) e^{i(x_m - x_n)p_A} dx_A \, dp_A \\
& \quad \times \iint \mathrm{Ai}\left(\frac{2x_B - x_n - x_m}{2^{1/3} x_0} + 2^{2/3} x_0^2 p_B^2\right) e^{i(x_n - x_m)p_B} dx_B \, dp_B \\
& + \iint x_A \mathrm{Ai}\left(2^{2/3}\left(\frac{x_A - x_m}{x_0} + x_0^2 p_A^2\right)\right) dx_A \, dp_A \\
& \quad \times \iint \mathrm{Ai}\left(2^{2/3}\left(\frac{x_B - x_n}{x_0} + x_0^2 p_B^2\right)\right) dx_B \, dp_B \Bigg\}.
\end{aligned} \tag{A.11}$$

Thus, using the results we have found for the integrals,

$$\langle x_A \rangle = \frac{A_n^2 A_m^2}{2\pi^2} \frac{x_0^2}{2^{2/3}} \left\{ -\frac{2^{4/3}\pi}{3A_n^2} \alpha_n \cdot \frac{2^{1/3}\pi}{x_0 A_m^2} + 0 + 0 - \frac{2^{4/3}\pi}{3A_m^2} \alpha_m \cdot \frac{2^{1/3}\pi}{x_0 A_n^2} \right\} \tag{A.12a}$$

$$= -\frac{x_0}{3} \left(\alpha_n + \alpha_m \right). \tag{A.12b}$$

Notice that the result is dimensionally correct, since the quantity x_0 has dimensions of length. Moreover, since all the roots α_n of the Airy function have negative values, the negative sign of Eq. (A.12b) ensures that the result is within the allowed range for the GQW, $x \geq 0$.

It should be said, however, that the case presented here is one of the simplest. When we proceed to the calculation of moments in p, p^2, and x^2, the cyclic property of Eq. (A.1) becomes much more relevant and needs to be used explicitly to handle results involving higher-order derivatives of the Airy function. The relationship of Eq. (A.2) also becomes quite useful when we are not able to rewrite the integrals in terms of tabulated results. For example, the relationships of Eqs. (3.4) and (3.5) in Chapter 3 can only be obtained through methods involving the integral formulation of the Airy function of Eq. (A.2). The same is also true for the integrals of type II with $O(x,p) = p$ and $O(x,p) = p^2$, whose results are exhibited in Table (3.1).